Math 131 B, Lecture 2
Analysis

Midterm 2

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

(a) [5pts.] Let (M, d_M) and (S, d_S) be two metric spaces. Give a definition of a continuous function $f : M \to S$.

> **Solution:** We say that f is continuous if for every $p \in M$ and $\epsilon > 0$ there exists $\delta > 0$ such that whenever $d_M(x, p) < \delta$, $d_S(x, p) < \epsilon$.

(b) [5pts.] For each of the following pairs of metric spaces, either construct a continuous function $f : M \to S$ with $f(M) = S$ or explain why one cannot exist.

- (a) $M = (3, 5)$, $S = \mathbb{Q}$.
- (b)$M = [0, 1]$, $S = C([0, 1] \to \mathbb{R})$.
- (c)$M = \mathbb{R}$, $S = \{(x, y) : x^2 + y^2 = 1\}$.
- (d)$M = (1, 2) \cup (3, 4)$, $S = \{0, 1, 2\}$.

> **Solution:** (a) M is connected and S is not, so there is no such map. (b) M is compact and S is not, so there is no such map. (c) $f(x) = (\cos x, \sin x)$. (d) M is compact and S is not, so there is no such map. (d) M has two components and S has three, so there is no such map.

Problem 2.

(a) [5pts.] Give a definition of a connected metric space M.

> **Solution:** We say that M is connected if there do not exist disjoint nonempty open sets A and B in M such that $M = A \cup B$.

(b) [4pts.] Prove that the intersection of two connected subsets of the real line is connected.

> **Solution:** Connected subsets of the real line are intervals. The intersection of two intervals is a (possibly trivial) interval.

(c) [1pts.] Give an example showing that the above result need not be true in an arbitrary metric space. (A sketch is fine, you don't need to prove it.)

> **Solution:** Consider two horseshoe shapes intersecting at their feet on the plane.

Problem 3.

(a) [5pts.] Let $\{f_n\}$ be a sequence of functions $f_n : S \to T$. What does it mean for f_n to converge uniformly to a function $f : S \to T$?

Solution: We say that $f_n \to f$ uniformly if for every $\epsilon > 0$ there is some N such that for $n > N$ and all $x \in S$, we have $d_T(f_n(x), f(x)) < \epsilon$.

(b) [5pts.] Prove that if $f_n \to f$ uniformly and each f_n is continuous, f is continuous.

Solution: Let $\epsilon > 0$, and let $x_0 \in S$. Choose N such that $d_T(f_N(x), f(x)) < \frac{\epsilon}{3}$ for all $x \in S$. Choose δ such that $d_S(x, x_0) < \delta$ implies that $d_T(f_N(x), f_N(x_0)) < \frac{\epsilon}{3}$. Then for $d_S(x, x_0) < \delta$, we have

$$\begin{aligned} d_T(f(x), f(x_0)) &\leq d_T(f(x), f_N(x)) + d_T(f_N(x), f_N(x_0)) + d_T(f_N(x_0), f(x_0)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &< \epsilon. \end{aligned}$$

Since x_0 was arbitrary, we conclude that $f : S \to T$ is continuous.

Problem 4.

(a) [5pts.] State Abel's Theorem.

Solution: If a power series $f(x) = \sum a_n(x-a)^n$ has radius of convergence R, and converges at one of its endpoints $x = a + R$ or $x = a - R$, then f is continuous at that endpoint. (Or, if $f(x) = \sum a_n x^n$ has radius of convergence 1 and $\sum a_n$ converges, $\lim_{x \to 1} f(x) = \sum a_n$.)

(b) [5pts.] Prove that the sum

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

converges to $\frac{\pi}{4}$.

Solution: We know that $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \int \sum_{n=0}^{\infty} \frac{(-x^2)^n}{2n} = \int \frac{1}{1+x^2} = \tan^{-1}(x)$ converges on $(-1, 1)$. When $x = 1$ we get the series in the statement of the question, which converges by the Alternating Series Test. So, by Abel's Theorem, the sum $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ is $\tan^{-1}(1) = \frac{\pi}{4}$.

Problem 5.

(a) [5pts.] State the Weierstrass M-test.

Solution: If $\{f_n\}$ is a sequence of real-valued functions, and for each n we have some $M_n > 0$ such that $|f_n(x)| < M_n$ for all $x \in S$, and moreover the series $\sum M_n$ converges, then f_n converges uniformly on S.

(b) [5pts.] Prove that the series $\sum_{n=2}^{\infty} \ln(1 + \frac{x}{n^2})$ converges uniformly on $(-1, 1)$. [Hint: How do the derivatives behave?]

Solution: Let $f_n = \ln(1 + \frac{x}{n^2})$. Notice that $f_n'(x) = \frac{1}{n^2} \cdot \frac{1}{1+\frac{x}{n^2}} = \frac{1}{n^2+x}$. Ergo $|f_n'(x)| \leq \frac{1}{n^2-1}$, so since $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges, by the Weierstrass M-test f_n' converges uniformly. Moreover, $\sum f_n(0) = \sum 0$ converges, so by our theorem concerning differentiation and uniform convergence, $\sum f_n$ converges uniformly on $(-1, 1)$.